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Field Equations of the Gauge Theory of Gravitation Originate from a Quadratic Lagrangian with Torsion

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The equations of the gauge theory of gravitation [Debney et al., *General Relativity and Gravitation*, **9**, 879–887 (1978)] are derived from a complex quadratic Lagrangian with torsion. The derivation is performed in a coordinate basis in a completely covariant way.

The purpose of this note is to briefly describe how the field equations of the gauge theory of gravitation (see, e.g., Debney et al., 1978) can be derived in a conceptually much simpler way from a quadratic Lagrangian with torsion. The present derivation requires only knowledge of standard tensor algebra and analysis; no familiarity with the gauge theory of gravitation is needed at all. The derivation also has the merit that it is generally covariant and can be performed in a coordinate (holonomic) basis, and that the equations are all derived from a Lagrangian, which is complex, however.

The equations as given by Debney et al. (1978) are

$$\zeta Q_{kl} + C_{kmln}Q^{mn} = 0 \qquad \text{[their equation (1.1)]} \tag{1}$$

$$^*C_{kmln}Q^{mn} = 0 \qquad \text{[their equation (1.3)]} \tag{2}$$

$$^*C_{mnrs;p}g^{mp} = 0 \qquad \text{[their equation (1.4)]} \tag{3}$$

The contraction of (1) yields $Q = 0$. As we derive the equations from a purely quadratic Lagrangian, it follows that $\zeta = 0$, and so our equations are more general in that they allow $Q \neq 0$.

We start with the Lagrangian density

$$L = \tfrac{1}{12}(-g)^{1/2}U_{xyuv}W^{xyuv} \tag{4}$$

Here,

$$U_{xyuv} = W_{xyuv} + M_{xyuv} \tag{5}$$

W_{xyuv} is a sort of a counterpart of the Weyl tensor in a manifold with torsion; it has the same symmetries

$$W_{xyuv} = W_{uvxy} = -W_{yxuv} = -W_{xyvu} \tag{6}$$

$$W_{x[yuv]} = 0 \tag{7}$$

It is related to the (2,0)+(0,2) representation of the Lorentz group [compare Hayashi 1968, equation (3.24) or Hayashi and Bregman 1973, Appendix IIb]; it is expressed as

$$W_{xyuv} = 2D_{xyuv} + 2D_{uvxy} + D_{vxyu} + D_{yuvx} - D_{vyxu} - D_{xuvy} \tag{8}$$

where

$$D_{xyuv} = R_{xyuv} + \tfrac{1}{6}R(g_{xu}g_{yv} - g_{xv}g_{yu}) - \tfrac{1}{2}(R_{yv}g_{xu} + R_{xu}g_{yv} - R_{yu}g_{xv} - R_{xv}g_{yu}) \tag{9}$$

Thus

$$D_{xyuv}g^{yv} = 0, \qquad W_{xyuv}g^{yv} = 0 \tag{10}$$

Here, R_{xyuv} is the Riemann tensor, which is the sum of the curvature tensor Q_{xyuv} and the distortion tensor P_{xyuv} (compare Gogala, 1980), which are defined as

$$Q_{xyuv} = \hat{\Gamma}_{xyv/u} - \hat{\Gamma}_{xyu/v} + \hat{\Gamma}^m{}_{yu}\hat{\Gamma}_{mxv} - \hat{\Gamma}^m{}_{yv}\hat{\Gamma}_{mxu} \quad \text{(in a coordinate basis)} \tag{11}$$

$$P_{xyuv} = S_{vxy;u} - S_{uxy;v} + (S_{vxm}S_{uyn} - S_{vym}S_{uxn})g^{mn} \tag{12}$$

Here, S_{uxy} denotes the contortion tensor, antisymmetric in the last two indices, $\hat{\Gamma}_{xyu}$ is the Levi-Cività connection, and ";" denotes the covariant derivative with respect to it.

Similarly to (8) and (9), we define

$$M_{xyuv} = 2E_{xyuv} + 2E_{uvxy} + E_{vxyu} + E_{yuvx} - E_{vyxu} - E_{xuvy} \tag{13}$$

where

$$E_{xyuv} = {}^*R_{xyuv} + \tfrac{1}{6}{}^*R(g_{xu}g_{yv} - g_{xv}g_{yu}) - \tfrac{1}{2}({}^*R_{yv}g_{xu} + {}^*R_{xu}g_{yv} - {}^*R_{yu}g_{xv} - {}^*R_{xv}g_{yu}) \tag{14}$$

The asterisk denotes the dual tensor; we have

$$^*R_{xyuv} = R_{kluv}\eta_{mnxy}g^{km}g^{ln} \tag{15}$$

with $^*R_{xu}$ and $^*R$ being its contractions over y and v, and x and u, y and v, respectively.

Note that

$$^*D_{xyuv} \neq E_{xyuv}, \qquad ^*W_{xyuv} \neq M_{xyuv} \tag{16}$$

We use the imaginary Levi-Cività tensor, defined as

$$\eta_{klmn} = \tfrac{1}{2}(+g)^{1/2}\varepsilon_{klmn}, \qquad \eta^{klmn} = \tfrac{1}{2}\frac{1}{(+g)^{1/2}}\varepsilon^{klmn} \tag{17}$$

so that the double dual of a tensor is equal to the original tensor itself, without any change of the sign.

The Lagrangian (4) is thus complex. It has an interesting property that the contortion tensor appears in it only in the self-dual combination

$$S_{jkl} + S_j{}^{mn}\eta_{mnkl} \tag{18}$$

As η_{mnkl} is imaginary, this combination recalls somewhat the (1,0) representation of the Lorentz group.

As the basic variables, with respect to which we perform the Hamiltonian differentiation, we take only tensor quantities; in our case, they are the components of the metric and the contortion tensors.

The Hamiltonian derivative of (4) with respect to g_{ab} is[1]

$$G_{ab} = \tfrac{1}{12} G_{ab} W_{xyuv} W_{xyuv} - \tfrac{2}{3} W_{xyua} W_{xyub} + X_{ab}$$
$$- 2U_{xaub;u;x} - 2U_{xbua;u;x} + U_{xauv} Q_{xbuv} + U_{xbuv} Q_{xauv}$$
$$+ 2U_{xyuv}\left(S_{vya} S_{uxb} + S_{vyb} S_{uxa}\right) + 2\left[\left(U_{xaub} + U_{xbua}\right) S_{uxz}\right]_{;z}$$
$$+ 2\left[U_{xyub}\left(S_{uya} + S_{yua}\right)\right]_{;x} + 2\left[U_{xyua}\left(S_{uyb} + S_{yub}\right)\right]_{;x} \tag{19}$$

It is of course symmetric in a and b.

Here

$$X_{ab} = - {}^*R_{xyub}\left(R_{xyua} + R_{uaxy} + R_{axyu} + R_{yuax}\right)$$
$$- {}^*R_{xyua}\left(R_{xyub} + R_{ubxy} + R_{bxyu} + R_{yubx}\right)$$
$$+ 3\,({}^*R_{xu} + {}^*R_{ux})\left(R_{xaub} + R_{xbua}\right) + 3\,{}^*R_{xa}\left(R_{xb} + R_{bx}\right)$$
$$+ 3\,{}^*R_{xb}\left(R_{xa} + R_{ax}\right) - 2\,{}^*R\left(R_{ab} + R_{ba}\right) \tag{20}$$

Note also, that W_{xyuv} satisfies the conditions for the Bach–Lanczos–Lovelock identity (cf., e.g., Lovelock and Rund, 1975, p. 128, exercise 4.9, and p. 293, exercise 7.37), so that the first term in (19) is equal to

$$\tfrac{1}{3} W_{xyua} W_{xyub} \tag{21}$$

The Hamiltonian derivative with respect to S_{abc} is antisymmetric in b and c, and has the form

$$A_{abc} = 4\left(U_{bcaz;z} - U_{xbua} S_{uxc} + U_{xcua} S_{uxb}\right) \tag{22}$$

An expansion into components shows that its real and imaginary parts are related by the proportionality relation

$$A_{(r)abc} - {}^*A_{(i)abc} = A_{(i)a}{}^{de}\eta_{debc} \tag{23}$$

[1]In the equations (19), (20), (22), (24), (26), (28)–(36), (38), and (39), we write all the contravariant indices as covariant indices in order to enhance the lucidity of the equations and to simplify the printing. The interpretation is simple; if a covariant index appears twice, that means that one of the two indices is a contravariant index. We can afford to do that because all the quantities involved are either tensors or (metric-preserving) covariant derivatives of tensors.

Thus, (22) yields only 24 independent field equations of the form

$$A_{(r)abc} = 4(W_{bcaz;z} - W_{xbua}S_{uxc} + W_{xcua}S_{uxb}) = 0 \tag{24}$$

When the contortion tensor is identically zero, then (8) is proportional to the classical Weyl tensor

$$W_{xyuv}(S_{vxy} \equiv 0) = 6C_{xyuv} \tag{25}$$

and so

$$A_{(r)abc}(S_{vxy} \equiv 0) = 24C_{bcaz;z} = 0 \tag{26}$$

This system of equations is identical to (3), because the left and the right duals of the Weyl tensor are equal to each other.

Instead of requiring that G_{ab} (19) be zero, we can now require, thanks to (24) and (23), that the combination

$$\begin{aligned} H_{ab} = G_{ab} &+ \tfrac{1}{2}(A_{abc;d} + A_{bac;d})g^{cd} \\ &+ \tfrac{1}{2}(A_{mbc}S^{m}{}_{ad} + A_{mac}S^{m}{}_{bd})g^{cd} \end{aligned} \tag{27}$$

be zero. By inserting (22) into (27) we find

$$H_{ab} = (-\tfrac{1}{3})W_{xyua}W_{xyub} + X_{ab} + U_{xauv}R_{xbuv} + U_{xbuv}R_{xauv} \tag{28}$$

An explicit calculation shows that $H_{ab}g^{ab} \equiv 0$, so that (28) results in only nine independent complex equations. They split into nine real equations,

$$H_{(r)ab} = W_{xauv}(R_{xbuv} - \tfrac{1}{6}W_{xbuv}) + W_{xbuv}(R_{xauv} - \tfrac{1}{6}W_{xauv}) = 0 \tag{29}$$

and nine imaginary equations,

$$H_{(i)ab} = X_{ab} + M_{xauv}R_{xbuv} + M_{xbuv}R_{xauv} = 0 \tag{30}$$

For zero contortion, (29) becomes

$$H_{(r)ab} = 6C_{xauv}(Q_{xbuv} - C_{xbuv}) + 6C_{xbuv}(Q_{xauv} - C_{xauv}) = 0 \tag{31}$$

which after the insertion of the explicit expression for C_{xyuv} yields

$$H_{(r)ab} = 12C_{xaub}Q_{xu} = 0 \tag{32}$$

which is equation (1) with $\zeta = 0$.

Similarly, $H_{(i)ab}$ becomes for zero contortion

$$H_{(i)ab} = 3\left[-2Q_{klxa}Q_{mnxb}\eta_{klmn} + Q_{mnxy}\left(Q_{alxy}\eta_{mnbl} + Q_{blxy}\eta_{mnal}\right)\right]$$
$$= 3\left[\left(-Q_{klxb} + {}^*Q^*_{klxb}\right)Q_{mnxa} + \left(-Q_{klxa} + {}^*Q^*_{klxa}\right)Q_{mnxb}\right]\eta_{klmn} \tag{33}$$

An expansion of the double dual gives

$$^*Q^*_{klxb}Q_{mnxa}\eta_{klmn} = Q_{klxb}Q_{mnxa}\eta_{klmn} + 2Q_{xu}Q_{mnxa}\eta_{mnub} \tag{34}$$

Thus

$$H_{(i)ab} = 6Q_{xu}\left({}^*Q_{ubxa} + {}^*Q_{uaxb}\right) = 12Q_{xu}{}^*Q_{xaub} = 12Q_{xu}{}^*C_{xaub} = 0 \tag{35}$$

which is equation (2).

It has been shown by (Debney et al., 1978) that the system of equations (1), (2), and (3), have the same solutions as Einstein equations in vacuum, $Q_{ab} = 0$. The system of equations (32), (35), and (26) may allow more general solutions, because the restriction $Q = 0$ is not there any more. The system of equations (29), (30), and (24) may even have nontrivial solutions, which are regular everywhere thanks to additional degrees of freedom, offered by the components of the contortion tensor. The details of these considerations and more details of the calculations leading to the results of the present paper will be published separately.

By combining (26) with (3), we get

$$C_{bcad;z;z} = +C_{kcdz}Q_{kbaz} + C_{bkdz}Q_{kcaz} + C_{bckz}Q_{kdaz} + C_{bcdk}Q_{ka}$$
$$- C_{kcaz}Q_{kbdz} - C_{bkaz}Q_{kcdz} - C_{bckz}Q_{kadz} - C_{bcak}Q_{kd} \tag{36}$$

For sufficiently small curvatures, this is the wave equation for Weyl tensor.

Although we now have new field equations, we can consider that Einstein equations for matter are still valid, but not as field equations. They can rather be considered as definition equation for the matter tensor in terms of the space-time curvature (Eddington and Schrödinger interpretation), that is

$$T_{ik} = Q_{ik} - \tfrac{1}{2}g_{ik}Q \tag{37}$$

rather than the other way around. As the equations (32) and (35) can also be written as

$$(Q_{xu} - \tfrac{1}{2}g_{xu}Q)C_{xa\,ub} = T_{xu}C_{xa\,ub} = 0 \tag{38}$$

$$(Q_{xu} - \tfrac{1}{2}g_{xu}Q)^{*}C_{xa\,ub} = T_{xu}{}^{*}C_{xa\,ub} = 0 \tag{39}$$

they can be interpreted as describing the coupling between the gravitating matter and the pure gravitational field.

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